OVERBOOKING MODELS FOR HOTEL REVENUE MANAGEMENT

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Abstract

Revenue Management (RM) is an essential strategy for industries having fixed perishable resources to maximize overall revenue by allocating the available resources to the right customers at the right price at the right time. Overbooking is one of the RM tools for hotel revenue management that helps a hotel manager to decide when and how much extra number of rooms to sell. Overbooking arises as some of the customers may not eventually show up (so called “no show”) on the booking date, while others may cancel or amend their bookings in the last minutes. The objective of this paper is to develop overbooking models to determine the optimal number of overbooking for hotels having one and two different types of rooms. We prove that for hotels with only one type of room, there exists a closed form solution to guarantee the optimal number of overbooking, depending on the cost of walking customers to other hotels, the cost of unsold rooms and cancelation distribution observed in the past. For hotels with two types of room, we prove the convexity structure and identify equations to seek the number of overbooking for low-price and high-price rooms. We also provide key comparative statics on how model parameters impact the optimal decisions under different scenarios.

Keywords: Overbooking, Hotel revenue management, Inventory management

1. INTRODUCTION

Revenue Management is an essential strategy to maximize revenue for industries with limited resources. Revenue Management (defined shortly as “RM”) is extensively applied to manage consumer behaviors (Anderson and Xie, 2010), by utilizing predictable information to calculate appropriate price and time to sell resources for different sections of customers (Haddad et al., 2008). Revenue management was pioneered in the airline industry, and then it was widely practiced in other industries such as hotels, cruise lines and rental cars, etc. (Cross, 1997).

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Hotel industry is one of business sectors that has grown continuously every year with intensive competition. Hotels can benefit from RM strategy to increase revenue by balancing their demand and using variable prices, that is selling the right room to the right person at the right time for the right price (Talluri and Ryzin, 2005). An example of hotel RM can be presented as a system, illustrated in Figure 1. The process starts when a customer places a booking request, which is recorded via the hotel's revenue management system. This functional system consists of four basic components: data and information, hotel revenue centre, RM software and RM tools. All elements collaborate together to create the determinate process of the RM optimization. The operational results from the RM process, expressed in Figure 1, are the number of rooms, category of rooms, the specific booking element from the booking request status (confirmed/rejected), price rates, duration of stay, cancellation and amendment conditions. Afterwards, the hotel management team can consider all results to make a proper decision; to handle the demand of customer's reservation.

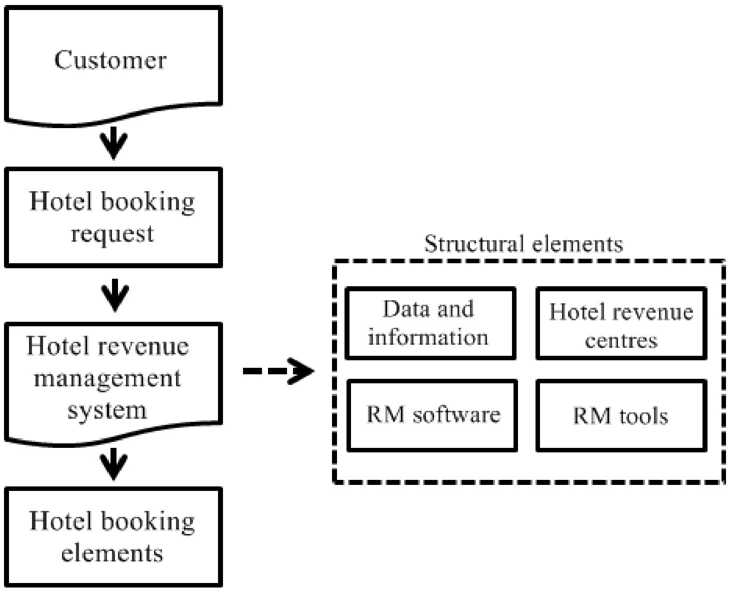

Figure 1: Components of hotel revenue management system.

For service industry like hotels, reservations are placed in advance and the hotel manager must cope with the problem of no-shows, i.e., customers fail to show up after making a reservation. It has been revealed that even if the hotel is assured of payment and there are penalties for oversales, the walk-in bookings cannot overcome the loss of reservations from late cancellations and no-shows. A method that hotels can reduce their costs due to revenue-losing no-shows is to allow overbooking, which is one of the revenue management tools. It was found in the literature that if hotels can well manage the overbooking system, overbooking can help increase approximately 20% of the total revenues (Vinod, 1992).

Overbooking means the hotel accepts customer's reservations more than the number of existent rooms. And then they expect that the quantity of overbooking will equal the number of canceling or amending rooms in the last minutes. Therefore, the hotel management team must carefully plan and calculate the right room units to allow overbooking (Hadjinicola and Panayi, 1997; Ivanov, 2006; Koide and Ishii, 2005; Netessine and Shumsky, 2002; Pullman and Rogers, 2010).

There have been several proposed approaches to calculate the optimal number of rooms to be allowed for overbooking. Netessine and Shumsky (2002) developed a method to find the number of rooms to be overbooked for one-type room hotel. However, their research did not consider various types of overbooking distribution and more complicated types of room cases. Ivanov (2006) proposed a simple method to find the optimal number of rooms to be overbooked for hotels with two different room types. Nevertheless, his decision model did not consider the marginal cost for each room unsold caused by no shows and the marginal cost for each walking guest. Thus, this paper's objective is to develop overbooking models for the hotels having both one and two different types of room. The objective cost function presented in this paper considers incurred costs from both leftover rooms and insufficient rooms in different manner as compared to existing papers proposed in the literature.

The goal of overbooking models presented in this paper is the find the optimal number of booking for each type of rooms that can minimize the total cost. The expectation of total cost consists of two main part:

1. The cost incurred for leftover hotel rooms caused by no shows or late cancellation. That is the cost of opportunity losing for rooms reserved but not occupied.
2. The cost incurred in case of having insufficient rooms (over-sales); the number of arrived reservations is greater than the number of rooms available.

This paper is organized as follows. In the next section, we present model description and formulations, as well as theoretical results of overbooking models for hotels with one and two types of rooms. Computational experiments on how solutions are affected by key model parameters are explored in Section 3. We conclude by summarizing important results and providing managerial insights in Section 4. The proofs of all results are provided in the Appendix.

2. OVERBOOKING MODELS

This section describes the idea of overbooking models development for hotels, having one type and two types of rooms. The main assumptions of this research are as follows:

1. The distribution of the customers who reserved but did not show up or made a late cancellation can be obtained from information in the past. Probability Density Function and Cumulative Density Function can be found accordingly.
2. If hotels do not have enough rooms for every customer that booked in, they can offer them upgraded rooms (rooms with higher price than what customers pay for). But if there is not enough room for every customer, hotels have ability to outsource the same level of rooms from other hotels nearby. The rooms replacement are as follows:
   - If a customer reserved a high-price room, but there are only low-price rooms left. The hotel will need to outsource a comparable room from associated hotels in the area.
   - If a customer reserved a low-price room, but there are no low-price rooms left. We can classify this situation in two cases below:
     - If there are some high-price rooms left, the hotel can upgrade and offer a high-price room to the customer.
     - If there are no available high-price rooms, the hotel will need to outsource a comparable room from associated hotels in the area.

3. When there are unsold rooms from no-show customers, the hotel misses opportunity to sell that room to other customers and it is considered as the loss of unsold rooms. When there are not enough rooms for every reserved customer (over-sales), the hotel needs to offer a deep discount to each customer sent to other hotels in the area. The loss in the later case is assumed to be higher than the first (loss of unsold rooms) as it is related to brand image and other difficulties of walking customers.

From the assumptions described above, we formulate overbooking models to determine the optimal number of rooms to be overbooked for hotels with one type and two types of rooms, respectively.

2.1 Hotels with one type of rooms

In this section, we present the overbooking model for hotels with only one type of rooms, i.e., all rooms are identically priced. Let X be the number of rooms cancelled or no show, $f(x)$ and $F(x)$ be the probability density function and the cumulative density function of X, respectively. Define r as the average loss of revenue per each room unsold each night that are caused by no show, and C as the average cost incurred for each room that the hotel need to send customers to other associated hotels nearby (over-sales). Let the decision variable, Q, be the number of the overbooking rooms.

The overbooking model considers the loss of revenue and cost from the following two possible cases: (1) unsold rooms, (2) over-sales. If the first case (unsold rooms) happens, it means the number of rooms reserved (but customers do not show up) is more than the number of overbooking rooms ($X > Q$). Let $TC_{X>Q}$ be the expectation of total loss from the case of unsold rooms $(X > Q)$ and it can be written as:

$$TC_{X>Q} = \int_{Q}^{\infty} r(x-Q)f(x)dx .$$

The second case (over-sales) means that the number of customers who reserved but did not shown up is less than or equal to the number of overbooking rooms $(X \le Q)$. Therefore, the hotel's expected total cost incurred from sending customers to other hotels nearby is:

$$TC_{X\le Q} = \int_{0}^{Q} C(Q-x)f(x)dx .$$

Let TC be the expectation of total loss of revenue and cost of over-sales incurred. It can be written as:

$$TC = TC_{X>Q} + TC_{X\le Q} = \int_{Q}^{\infty} r(x-Q)f(x)dx + \int_{0}^{Q} C(Q-x)f(x)dx .$$

Theorem 1. *The optimal number of the overbooking rooms (for hotels with one room type) can be determined by the following equation:*

$$F(Q^*) = \frac{r}{C+r} .$$

Theorem 1 provides an equation to find the optimal number of overbooking rooms, The proof of Theorem 1 (convexity property and the first order condition derivation) can be referred in the Appendix. It can be noticed that the optimal number of overbooking depends on the ratio of r and $C+r$. When C (the average cost incurred for each room over-sold that the hotel need to send customers to other hotels nearby) increases, it is more beneficial to reduce the number of overbooking rooms (Q^*). On the other hand, when r (the average loss of revenue per each unsold room caused by no show) increases, it is more beneficial to increase Q^*.

2.2 Hotels with two types of rooms

In this section, we present the overbooking model for hotels with two types of rooms:

high and low price. This usually happens for large hotels where they offer standard rooms with lower prices, as well as some deluxe rooms with higher prices. Let i represent the type of rooms ($i=1$ refers to low-price rooms and $i=2$ refers to high-price rooms). Let X_i be the number of rooms cancelled or no show for type-i rooms, $f_i(x_i)$ and $F_i(x_i)$ be the probability density function and the cumulative density function of X_i, respectively. Define r_i as the average loss of revenue per each type-i room that is unsold each night that are caused by no show, and C_i as the average cost incurred for each over-sold type-i room that the hotel need to send customers to other hotels nearby. Let the decision variable Q_i be the optimal number of the overbooking type-i rooms.

The overbooking model for two types of rooms considers the loss of revenue for unsold rooms as well as the cost of having over-sales. Having more than one type of rooms in this case, there are four possible situations for the values of X_1 , X_2 , Q_1 , Q_2 , as shown in Table 1.

Table 1: Four possible situations of two types of room

Situations	Values
1	$X_2 > Q_2$ and $X_1 > Q_1$
2	$X_2 > Q_2$ and $X_1 \le Q_1$
3	$X_2 \le Q_2$ and $X_1 > Q_1$
4	$X_2 \le Q_2$ and $X_1 \le Q_1$

In situation 1 and 2, $X_2 > Q_2$ which means there are some unsold high-price rooms left since the number of high -price rooms reserved and then cancelled (or no show) is larger than the number of high -price rooms the hotel overbooked. The expected loss to the hotel can be written as: $\int_{Q_2}^{\infty} r_2(x_2 - Q_2) f_2(x_2) dx_2$.

In situation 1, we also have $X_1 > Q_1$, which means there are some unsold low-price rooms since the number of low-price rooms reserved and then cancelled (or no show) is larger than the number of low-price rooms that the hotel overbooked The expected loss to the hotel is: $(1 - F_2(Q_2)) \int_{Q_1}^{\infty} r_1(x_1 - Q_1) f_1(x_1) dx_1$.

In situation 2, the high-price rooms are insufficient as $X_2 > Q_2$. There are two possible cases.

1) If the number of insufficient low-price rooms is less than the number of unsold high-price rooms, the hotel can offer upgrades for all remaining customers. This loss can be considered as upgrading cost to offer more expensive rooms than what customers pay for. The expected loss in this case is as followed:

$$\int_{Q_2}^{\infty}\int_{Q_2+Q_1-x_1}^{Q_1}(r_2-r_1)(x_2-Q_2-Q_1+x_1)f_1(x_1)f_2(x_2)dx_1dx_2$$

2) If the number of insufficient low-price rooms is more than the number of unsold high-price rooms. The hotel can offer high-priced rooms to some customers until there are no available high-price rooms left. Then the hotel will need to outsource comparable rooms from associated hotels in the area to the rest of customers. The expected outsource cost can be computed by:

$$\int_{Q_2}^{Q_1+Q_2}\int_{0}^{Q_1+Q_2-x_2}C_1(Q_1-x_1-x_2+Q_2)f_1(x_1)f_2(x_2)dx_1dx_2\ .$$

Let $TC_{[j]}$ be the expectation of total cost from situation j. The expectation of total cost for situation 1 and 2 can be summarized as:

$$\begin{aligned}TC_{[1]+[2]} &= \int_{Q_2}^{\infty} r_2(x_2-Q_2)f_2(x_2)dx_2+(1-F_2(Q_2))\int_{Q_1}^{\infty} r_1(x_1-Q_1)f_1(x_1)dx_1\\ &+\int_{Q_2}^{\infty}\int_{Q_2+Q_1-x_1}^{Q_1}(r_2-r_1)(x_2-Q_2-Q_1+x_1)f_1(x_1)f_2(x_2)dx_1dx_2\\ &+\int_{Q_2}^{Q_1+Q_2}\int_{0}^{Q_1+Q_2-x_2}C_1(Q_1-x_1-x_2+Q_2)f_1(x_1)f_2(x_2)dx_1dx_2\ .\end{aligned}$$

For situation 3 and 4, the high-price rooms are insufficient, since the number of high-price rooms reserved and then cancelled (or no show) is less than or equal to the number of high-price rooms overbooked ($X_2 \le Q_2$). Therefore, the hotel needs to send customers to stay in the associated hotels instead, no matter they have low-price rooms left or not. The expected loss to the hotel can be written as: $\int_{0}^{Q_2} C_2(Q_2-x_2)f_2(x_2)dx_2$.

For situation 3, there are some unsold low-price rooms. Thus its expected loss of revenue can be written as: $F_2(Q_2)\int_{Q_1}^{\infty} r_1(x_1-Q_1)f_1(x_1)dx_1$.

And for situation 4, the low-price rooms are insufficient. However, there are also no high-price rooms left. So remaining customers booked for low-price rooms need to be sent to other associated hotels nearby. The expected cost from outsourcing the low-price rooms is: $F_2(Q_2)\int_{0}^{Q_1} C_1(Q_1-x_1)f_1(x_1)dx_1$.

Thus, the expectation of total cost for situation 3 and 4 can be summarized as:

$$TC_{[3]+[4]} = \int_{0}^{Q_2} C_2(Q_2-x_2)f_2(x_2)dx_2 + F_2(Q_2)\int_{Q_1}^{\infty} r_1(x_1-Q_1)f_1(x_1)dx_1 + F_2(Q_2)\int_{0}^{Q_1} C_1(Q_1-x_1)f_1(x_1)dx_1$$

Considering all situations (1 to 4), the expectation of total cost is equal to:

$$\begin{aligned}
TC &= TC_{[1]+[2]} + TC_{[3]+[4]} \\
&= \int_{Q_2}^{\infty} r_2(x_2-Q_2)f_2(x_2)dx_2 + (1-F_2(Q_2))\int_{Q_1}^{\infty} r_1(x_1-Q_1)f_1(x_1)dx_1 \\
&+ \int_{Q_2}^{\infty}\int_{Q_2+Q_1-x_1}^{Q_1} (r_2-r_1)(x_2-Q_2-Q_1+x_1)f_1(x_1)f_2(x_2)dx_1dx_2 \\
&+ \int_{Q_2}^{Q_1+Q_2}\int_{0}^{Q_1+Q_2-x_2} C_1(Q_1-x_1-x_2+Q_2)f_1(x_1)f_2(x_2)dx_1dx_2 \\
&+ \int_{0}^{Q_2} C_2(Q_2-x_2)f_2(x_2)dx_2 + F_2(Q_2)\int_{Q_1}^{\infty} r_1(x_1-Q_1)f_1(x_1)dx_1 + F_2(Q_2)\int_{0}^{Q_1} C_1(Q_1-x_1)f_1(x_1)dx_1 .
\end{aligned} \tag{1}$$

Lemma 1. *The total cost function presented in equation (1) is jointly convex with respect to Q_1 and Q_2.*

Theorem 2. *The optimal number of the overbooking* for low-price and high-price rooms *(Q_1* and Q_2*) can be determined by the following two equations:*

$$0 = C_1F_1(Q_1)F_2(Q_2) - r_1(1-F_1(Q_1)) + C_1\int_{Q_2}^{Q_2+Q_1} F_1(Q_2+Q_1-x_2)f_2(x_2)dx_2$$

$$+(r_2-r_1)[\int_{Q_2}^{\infty} F_1(Q_2+Q_1-x_2)f_2(x_2)dx_2 - F_1(Q_1)(1-F_2(Q_2)) + f_1(Q_1)\int_{Q_2}^{\infty}(x_2-Q_2)f_2(x_2)dx_2]$$

and

$$0 = C_2F_2(Q_2) - r_2(1-F_2(Q_2)) + (r_2-r_1)[\int_0^{Q_1} f_1(x_1)F_2(Q_2+Q_1-x_1)dx_1 - F_1(Q_1)]$$

$$+C_1[\int_0^{Q_1} f_1(x_1)F_2(Q_2+Q_1-x_1)dx_1 - F_1(Q_1)F_2(Q_2)].$$

The proof of Lemma 1 and Theorem 2 can be referred in Appendix. Theorem 2 provides two equations for determining the optimal number of overbooking for low-price and high-price rooms. The solutions of this model depend on model parameters, e.g., the distributions of number of rooms cancelled or no show for each room type, the average loss of revenue per each room type, the average cost of over-sales for each room type. We explore behaviors of the optimal solutions via computational experiments described in the next section.

3. COMPUTATIONAL EXPERIMENTS

In this section, we perform computational experiments for the overbooking model of hotels with two types of rooms with a goal of understanding the properties of the optimal solutions and the model performance under different scenarios. Specifically, we consider how model parameters, namely (1) the average loss of revenue per each type-i room unsold each night that are caused by no show and (2) the average over-sales cost incurred for each type-i room that the hotel need to send customers to other hotels nearby, affects the optimal number of the overbooking for each type of rooms.

Consider a hotel having 140 rooms in total (100 rooms for type-1 with low price and 40 rooms for type-2 with high price). Let X_1 ~ *uniform(0,50)*, and X_2 ~ *uniform(0,20).* Figure 2 shows the optimal number of the overbooking for type-1 and type-2 rooms at different values of the average loss of revenue per each type-1 room unsold each night that are caused by no show (r_1). We can notice that as the average loss of revenue per each type-1 room (low price) unsold each night increases, it is more beneficial to increase the number of overbooking for type-1 rooms (Q_1*). However, the increase of r_1 has only a little impact on the number of overbooking for type-2 rooms (Q_2*). It is more beneficial to increase Q_2* when r_1 increases, but as r_1 is larger and larger, it does not cause any change in Q_2* anymore. This is because when Q_1* increase due to the increase of r_1, it is more likely that for the hotel to over-sale type-1 rooms. Thus, it is also more likely for the hotel to upgrade customers to type-2 rooms (high price). The model, therefore, does not suggest any increase in Q_2* when Q_1* is already high.

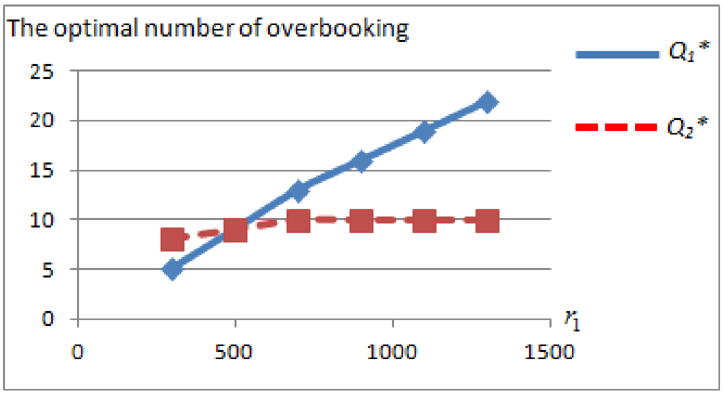

Figure 2: The optimal number of the overbooking for each type-1 and type-2 rooms at different values of r_1 (when r_2=1500, C_1=1000, C_2=2000)

Figure 3 shows the optimal number of the overbooking for type-1 and type-2 rooms at different values of the average loss of revenue per each type-2 room unsold each night that are caused by no show (r_2). We can notice that as the average loss of revenue per each type-2 room (high price) unsold increases, it is more beneficial to increase the number of overbooking for type-2 rooms (Q_2*). Nevertheless, as r_2 increases, the model will suggest the hotel manager to reduce the number of overbooking for type-1 rooms (Q_1*). An underlining reason is that when Q_2* is suggested to be increased, it is less likely to have type-2 rooms available. It is also less likely to be able to upgrade customers who book for a type-1 room to type-2 room when there are not enough type-1 rooms. For this reason, it is safer to reduce the number of overbooking for type-1 rooms in this case.

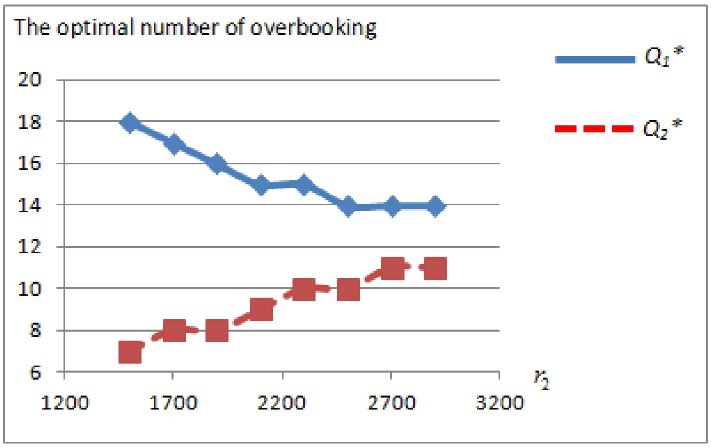

Figure 3: The optimal number of the overbooking for each type-1 and type-2 rooms at different values of r_2 (when r_1=500, C_1=1000, C_2=2000)

Next, we consider the effects of the average outsourcing cost incurred for each type-1 room over-sold and the hotel sends customers to other hotels nearby (C_1). Figure 4 presents the optimal number of the overbooking for each type of rooms at different values of C_1 .It can be observed that as the outsourcing cost for each type-1 room increases, it is more beneficial to reduce the number of overbooking for type-1 room since it is more costly for insufficient room situation. However, the increase in C_1 does not have any impacts on the optimal number of overbooking for type-2 room (with high price). It is because when the number of overbooking of low price (type-1) room decreases, it is less likely that the hotel will lack of type-1 rooms, so there are no needs to upgrade customers to type-2 rooms as well. The optimal number of overbooking for type-2 rooms is, therefore, not affected by the change of C_1.

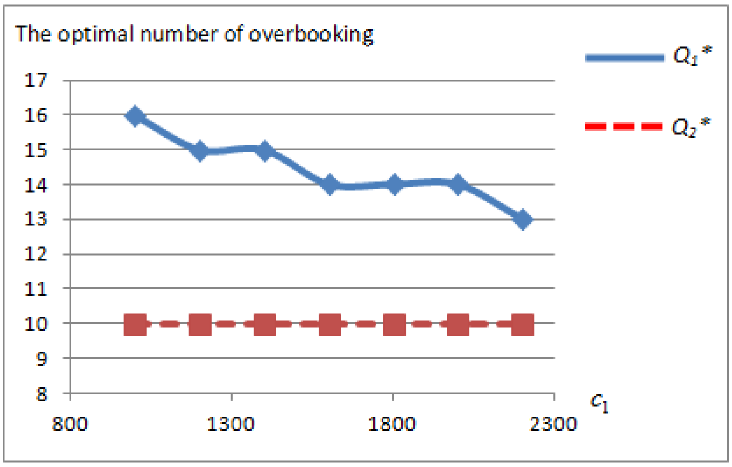

Figure 4: The optimal number of the overbooking for each type-1 and type-2 rooms at different values of C_1 (when r_1=500, r_2=1500, C_2=2000)

Figure 5 presents the optimal number of the overbooking for each type of rooms at different values of C_2 , i.e., the average outsourcing cost incurred for each type-2 room (over-sold) when the hotel sends customers to other hotels nearby. We can see that when it is more costly to outsource type-2 from other hotels, it is more beneficial to reduce the number of overbooking for type-2 rooms (Q_2*). Also, we observed the decrease in Q_1* when C_2 increases. A reason is that when there are not enough type-1 rooms, the hotel will consider upgrading customers to high-price rooms (type-2). If it is costly to outsource type-2 rooms, the hotel should reduce the insufficiency risk by reducing the number of overbooking for type-1 rooms as well.

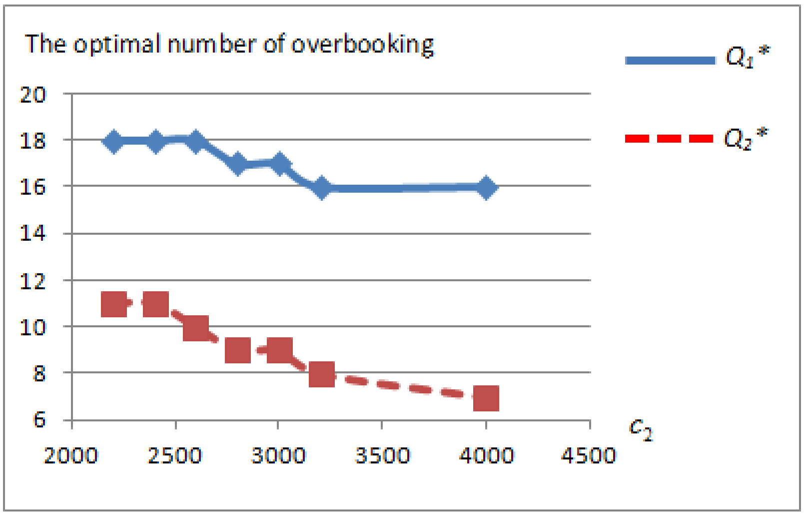

Figure 5: The optimal number of the overbooking for each type-1 and type-2 rooms at different values of C_2 (when r_1=500, r_2=1500, C_1=1000)

4. CONCLUSIONS

Overbooking decision is one of important and complicated decision makings, which are related directly to the yield of hotel revenue management. It is necessary for a hotel manager to observe cancellation pattern in the history to make a reliable decision. In this paper, we presented a systematic solution of the overbooking problem, by defining equations to identify the optimal number of overbooking for hotels having one and two room types. We showed that there exists a convexity structure of the objective cost function, consisting of the cost occurred from the leftover hotel rooms and the cost incurred from over-sales. Computational experiments are explored for insights on how model parameters affect the optimal number of the overbooking for each type of rooms. We found the number of the overbooking for each type-1 rooms with lower price should be reduced when the hotel manager observes the increase of (1) the average loss of revenue per each type-2 room unsold each night that are caused by no show, (2) the average outsourcing cost incurred for each type-1 room and (3) the average outsourcing cost incurred for each type-2 room when the hotel sends customers to other hotels nearby. The number of the overbooking for each type-2 rooms with lower price should be increased when the hotel manager observes the increase of the average loss of revenue per each type-1 or type-2 room unsold each night that are caused by no show. However the number of the overbooking for each type-2 rooms with lower price should be decreased when the average outsourcing cost incurred for each type-2 room over-sold increases.

There are several possible extensions of this paper. First, our model assumes a possible upgrade of rooms for customers. One can extend the model by allowing downgrades of rooms with some costs to see how it affects the optimal overbooking solutions for each type of rooms. A second possible extension is an overbooking model for hotels having three types of rooms. In this case there will be a greater number of possible situations to consider. It can be interesting to explore if there will be a convexity structure of the objective cost function in that case and how to identify the optimal solutions.

APPENDIX

Proof for Theorem 1.

The reduced form of the objective function can be written as:

$$TC = \int_{Q}^{\infty} r(x-Q)f(x)dx + \int_{0}^{Q} C(Q-x)f(x)dx\ .$$

By utilizing Leibniz Integral Rule, the objective function could be minimized at a relative ease. Taking the first order derivative on both sides of the objective function:

$$\frac{dTC}{dQ} = \frac{d}{dQ}[\int_{Q}^{\infty} r(x-Q)f(x)dx + \int_{0}^{Q} C(Q-x)f(x)dx] = CF(Q) - r(1-F(Q))\ .$$

Now, set the first derivative to zero to find the closed form expression for the overbooking level:

$$0 = CF(Q) - r(1-F(Q))\ .$$

Solving this for $F(Q^*)$ would give us:

$$F(Q^*) = \frac{r}{C+r}\ .$$

Now let us check the convexity of the objective function by taking the second derivative of the objective function.

$$\frac{d^2TC}{dQ^2} = (C+r)f(Q) > 0\ .$$

Since the second derivative of the function is nonnegative, our objective function is convex.

Proof for Lemma 1.

The convexity of the objective function can be proved by checking the second derivative of the objective function with respect to Q_1 and Q_2 respectively. We have:

$$\frac{\partial TC}{\partial Q_1} = C_1F_1(Q_1)F_2(Q_2) - r_1(1-F_1(Q_1)) + C_1\int_{Q_2}^{Q_2+Q_1} F_1(Q_2+Q_1-x_2)f_2(x_2)dx_2$$

$$+(r_2-r_1)[\int_{Q_2}^{\infty} F_1(Q_2+Q_1-x_2)f_2(x_2)dx_2 - F_1(Q_1)(1-F_2(Q_2)) + f_1(Q_1)\int_{Q_2}^{\infty}(x_2-Q_2)f_2(x_2)dx_2].$$

$$\frac{\partial TC}{\partial Q_2} = C_2F_2(Q_2) - r_2(1-F_2(Q_2)) + (r_2-r_1)(\int_0^{Q_1} f_1(x_1)F_2(Q_2+Q_1-x_1)dx_1 - F_1(Q_1))$$

$$+C_1(\int_0^{Q_1} f_1(x_1)F_2(Q_2+Q_1-x_1)dx_1 - F_1(Q_1)F_2(Q_2)).$$

Now, let us check the convexity of the objective function by taking the second derivative by Q_1 and Q_2 respectively of the objective function.

$$\frac{\partial^2 TC}{\partial Q_1^2} = r_1f_1(Q_1) + C_1f_1(Q_1)F_2(Q_2) + C_1\int_{Q_2}^{Q_2+Q_1} f_1(Q_2+Q_1-x_2)f_2(x_2)dx_2$$

$$+(r_2-r_1)(\int_{Q_2}^{\infty} f_1(Q_2+Q_1-x_2)f_2(x_2)dx_2 - f_1(Q_1)(1-F_2(Q_2)) + f_1'(Q_1)\int_{Q_2}^{\infty}(x_2-Q_2)f_2(x_2)dx_2)$$

$$\therefore \quad \frac{\partial^2 TC}{\partial Q_1^2} > 0$$

And; $$\frac{\partial TC}{\partial Q_1 \partial Q_2} = (r_2-r_1)(\int_{Q_2}^{\infty} f_1(Q_2+Q_1-x_2)f_2(x_2)dx_2 - f_1(Q_1)(1-F_2(Q_2))$$

$$+C_1\int_{Q_2}^{Q_2+Q_1} f_1(Q_2+Q_1-x_2)f_2(x_2)dx_2$$

$$\therefore \quad \frac{\partial^2 TC}{\partial Q_1^2} > \frac{\partial TC}{\partial Q_1 \partial Q_2}$$

And; $$\frac{\partial^2 TC}{\partial Q_2^2} = (r_2-r_1)\int_0^{Q_1} f_1(x_1)f_2(Q_2+Q_1-x_1)dx_1 + f_2(x_2)(r_2+C_2-C_1F_1(Q_1))$$

$$+C_1\int_0^{Q_1} f_1(x_1)f_2(Q_2+Q_1-x_1)dx_1$$

Since, $C_2 > C_1 \Rightarrow C_2F_1(Q_1) > C_1F_1(Q_1) \Rightarrow C_2 > C_2F_1(Q_1) > C_1F_1(Q_1) \Rightarrow C_2 > C_1F_1(Q_1)$

So, $\frac{\partial^2 TC}{\partial Q_2^2} > 0$

And, $$\frac{\partial TC}{\partial Q_2 \partial Q_1} = (r_2 - r_1)(\int_0^{Q_1} f_1(x_1) f_2(Q_2 + Q_1 - x_1) dx_1 - f_1(Q_1)(1 - F_2(Q_2)))$$

$$+C_1 \int_0^{Q_1} f_1(x_1) f_2(Q_2 + Q_1 - x_1) dx_1$$

$$\therefore \quad \frac{\partial^2 TC}{\partial Q_2{}^2} > \frac{\partial TC}{\partial Q_2 \partial Q_1}$$

Thus, we have:

$$(\frac{\partial^2 TC}{\partial Q_1{}^2})(\frac{\partial^2 TC}{\partial Q_2{}^2}) - (\frac{\partial TC}{\partial Q_1 \partial Q_2})(\frac{\partial TC}{\partial Q_2 \partial Q_1}) > 0 .$$

Since the second derivative of the function is nonnegative (By Hessian Matrix), our objective function is, therefore, jointly convex with respect to Q_1 and Q_2.

Proof for Theorem 2.

The the reduced form of the objective function can be written as:

$$TC = \int_{Q_2}^{\infty} r_2(x_2 - Q_2) f_2(x_2) dx_2 + (1 - F_2(Q_2)) \int_{Q_1}^{\infty} r_1(x_1 - Q_1) f_1(x_1) dx_1$$

$$+\int_{Q_2}^{\infty} \int_{Q_2+Q_1-x_1}^{Q_1} (r_2 - r_1)(x_2 - Q_2 - Q_1 + x_1) f_1(x_1) f_2(x_2) dx_1 dx_2$$

$$+\int_{Q_2}^{Q_1+Q_2} \int_{0}^{Q_1+Q_2-x_2} C_1(Q_1 - x_1 - x_2 + Q_2) f_1(x_1) f_2(x_2) dx_1 dx_2$$

$$+\int_0^{Q_2} C_2(Q_2 - x_2) f_2(x_2) dx_2 + F_2(Q_2) \int_{Q_1}^{\infty} r_1(x_1 - Q_1) f_1(x_1) dx_1$$

$$+F_2(Q_2) \int_0^{Q_1} C_1(Q_1 - x_1) f_1(x_1) dx_1 .$$

Applying Leibniz Integral Rule, the objective function could be minimized at a relative ease. Taking the first order derivative by Q_1 and Q_2 respectively on both sides of the objective function, we have:

$$\frac{\partial TC}{\partial Q_1} = C_1F_1(Q_1)F_2(Q_2) - r_1(1 - F_1(Q_1)) + C_1 \int_{Q_2}^{Q_2+Q_1} F_1(Q_2 + Q_1 - x_2)f_2(x_2)dx_2$$

$$+(r_2 - r_1)[\int_{Q_2}^{\infty} F_1(Q_2 + Q_1 - x_2)f_2(x_2)dx_2 - F_1(Q_1)(1 - F_2(Q_2)) + f_1(Q_1)\int_{Q_2}^{\infty}(x_2 - Q_2)f_2(x_2)dx_2].$$

$$\frac{\partial TC}{\partial Q_2} = C_2F_2(Q_2) - r_2(1 - F_2(Q_2)) + (r_2 - r_1)(\int_0^{Q_1} f_1(x_1)F_2(Q_2 + Q_1 - x_1)dx_1 - F_1(Q_1))$$

$$+C_1(\int_0^{Q_1} f_1(x_1)F_2(Q_2 + Q_1 - x_1)dx_1 - F_1(Q_1)F_2(Q_2)).$$

Now, set the first derivative to zero for two-equations to find the closed form expression for the overbooking level. Solving this for $F(Q_1^*)$ and $F(Q_2^*)$ would give us:

$$0 = C_1F_1(Q_1)F_2(Q_2) - r_1(1 - F_1(Q_1)) + C_1 \int_{Q_2}^{Q_2+Q_1} F_1(Q_2 + Q_1 - x_2)f_2(x_2)dx_2$$

$$+(r_2 - r_1)(\int_{Q_2}^{\infty} F_1(Q_2 + Q_1 - x_2)f_2(x_2)dx_2 - F_1(Q_1)(1 - F_2(Q_2)) + f_1(Q_1)\int_{Q_2}^{\infty}(x_2 - Q_2)f_2(x_2)dx_2).$$

And; $$0 = C_2F_2(Q_2) - r_2(1 - F_2(Q_2)) + (r_2 - r_1)(\int_0^{Q_1} f_1(x_1)F_2(Q_2 + Q_1 - x_1)dx_1 - F_1(Q_1))$$

$$+C_1(\int_0^{Q_1} f_1(x_1)F_2(Q_2 + Q_1 - x_1)dx_1 - F_1(Q_1)F_2(Q_2)).$$

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